Applied Time Series Topics

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Overview

1. Non-stationary data and consequences
2. Trends and seasonal cycles
3. Time series forecasting

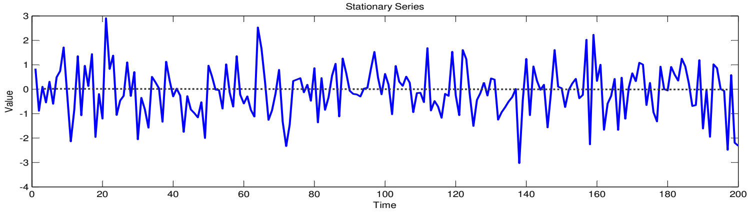
1. Stationarity and Non-Stationary Series

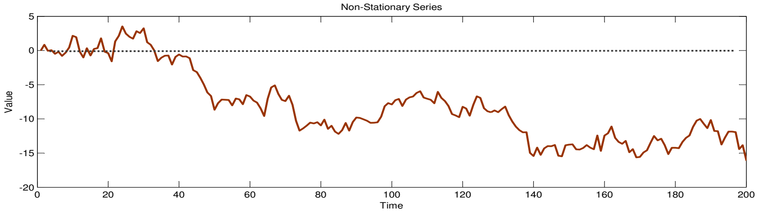
Stationarity

- A series is **stationary** if there is no systematic change in mean & variance over time
- *Example*: radio static
- A series is **non-stationary** if mean & variance change over time
- *Examples*: GDP, population, weather, etc.

1. Stationarity and Non-Stationary Series

Stationary vs Non-Stationary Series





1. Stationarity and Non-Stationary Series

Estimation Using Stationary Data

- Usual econometric techniques work (OLS, t-tests)
- Under usual assumptions, unbiased & efficient estimates

Estimation Using Non-Stationary Data

- Spurious regression results (biased estimates)
- Exceptionally high R^2 values and t-ratios
- No economic meaning

2. Testing for Non-Stationarity

Formally

- Augmented DickeyFuller test

Informally

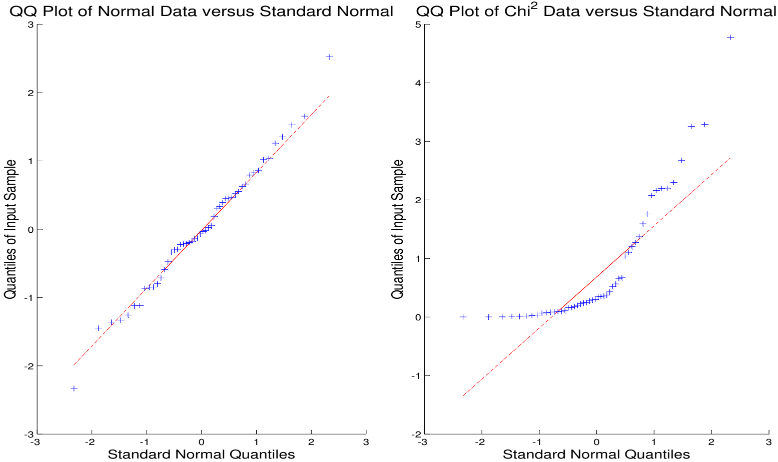
- Auto-Correlation Function (ACF)
- Normal Quantile Plot (Q-Q Plot)

2. Testing for Non-Stationarity

Quantile (Q-Q) Plot

- Generally: non-stationarity often leads to non-normal residuals
- Key idea: compare distribution of the residuals to normal
- Q-Q plot: scatter plot of residual quantiles against normal
  - **Stationary data**: quantiles match normal (45° line)
  - **Non-Stationary data**: quantiles don't match (points off 45° line)

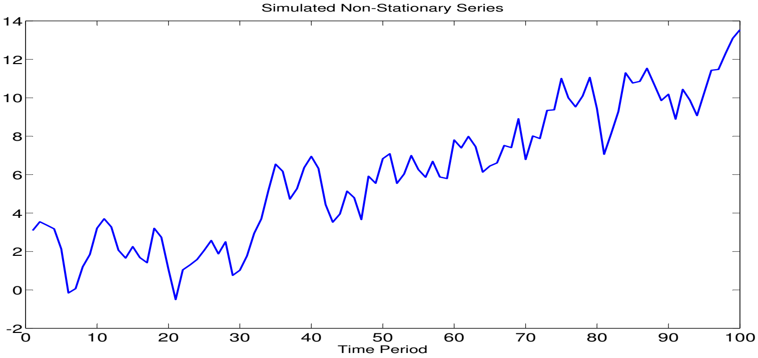
2. Testing for Non-Stationarity



2. Testing for Non-Stationarity

Example: Q-Q Residuals Plot (Non-Stationary Data)

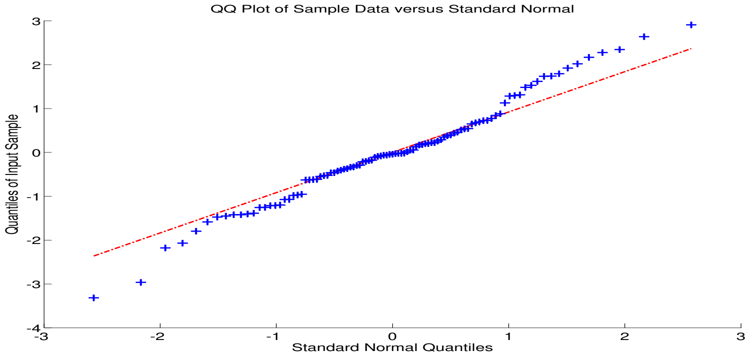
- Suppose: regress non-stationary series on its lag



2. Testing for Non-Stationarity

Example: Q-Q Residuals Plot (Non-Stationary Data)

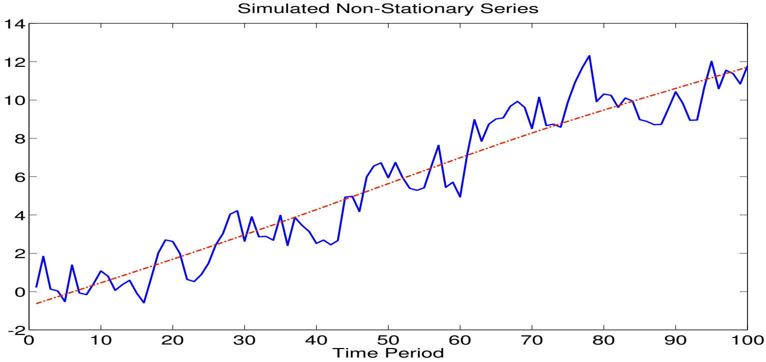
- Result: non-normal regression residuals (indicating problems)



3. Dealing With Non-Stationarity

Linear Trends

- Note that the trend in this example appears linear



3. Dealing With Non-Stationarity

Linear Trends

- Another way of dealing with trend is to **difference** the series
- $\Delta Y_t = Y_t - Y_{t-1}$ be the **first difference** of the series
- Then, estimate the model using first differences as

$$\Delta Y_t = \beta_1 + \beta_2 \Delta Y_{t-1} + u_t$$

- If first differences are non-stationary, use **second difference** $\Delta^2 Y_t = \Delta Y_t - \Delta Y_{t-1}$ and estimate

$$\Delta^2 Y_t = \beta_1 + \beta_2 \Delta^2 Y_{t-1} + u_t$$

3. Dealing With Non-Stationarity

Linear Trends

- One way to **deal with linear trend** is to include a **trend term**
- Instead of estimating a plain $AR(1)$ model

$$Y_t = \beta_1 + \beta_2 Y_{t-1} + u_t,$$

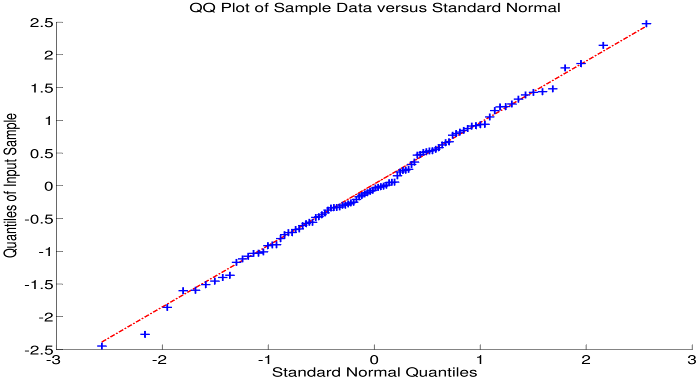
- Include time t into the regression & estimate

$$Y_t = \beta_1 + \beta_2 Y_{t-1} + \beta_3 t + u_t.$$

3. Dealing With Non-Stationarity

Q-Q Residuals Plot (Stationary Data)

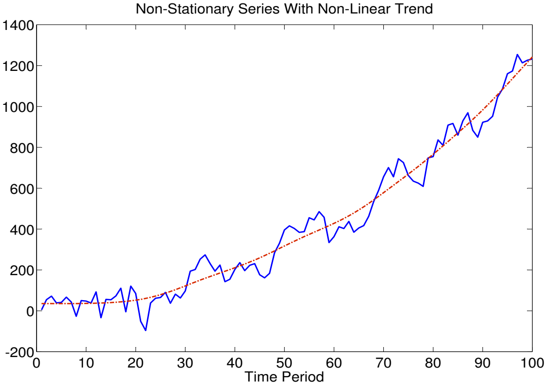
- Normal residuals once trend is successfully netted



3. Dealing With Non-Stationarity

Non-Linear Trends

- In some series **non-linear trends** may be present
- For example, population, output, may grow exponentially



3. Dealing With Non-Stationarity

Non-Linear Trends

- Non-linear trends can be dealt with by differencing
- Alternatively, include an **exponential time term**

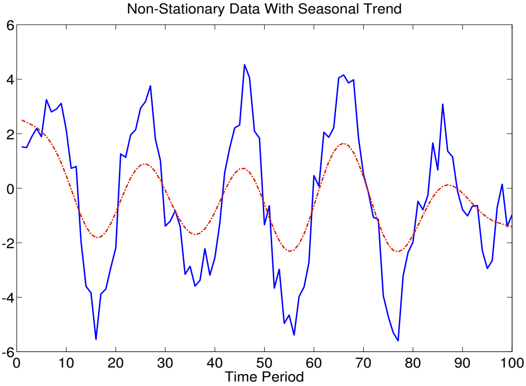
$$Y_t = \beta_1 + \beta_2 Y_{t-1} + \beta_3 t^{\beta_4} + u_t$$

- Residual Q-Q plot can be used to check model fit

3. Dealing With Non-Stationarity

Seasonal Trends

- Some series may exhibit **seasonal trends**
- For example, weather patterns, employment, inflation, etc.



3. Dealing With Non-Stationarity

Seasonal Trends

- Inclusion of linear or quadratic trend may be insufficient
- Several approaches to accounting for seasonal trends
  - Differencing
  - Modelling cyclical trends

3. Dealing With Non-Stationarity

Seasonal Differences

- Suppose trend cycle is repeated with frequency s periods
- For example, for monthly data
  - Annual cycles $s = 12$
  - Quarterly cycles $s = 3$
- Solution: work with **seasonal differences** $\Delta_t^s Y_t$

$$\Delta_t^s Y_t = Y_t - Y_{t-s}$$

- Examine the residual Q-Q plot to check model fit
- Choice of s may be challenging (experimentation)

3. Dealing With Non-Stationarity

Seasonal Trend Models

- As with linear or exponential trends, can explicitly include seasonal trend term into the model
- A common approach is to include **cyclical trend term** based on sine wave

$$\beta \sin \omega(t + \theta),$$

  where

  - t - time
  - β - cycle amplitude
  - ω - cycle length
  - θ - phase angle (starting phase)

3. Dealing With Non-Stationarity

Seasonal Trend Models

- Include cyclical trend term into the model by estimating

$$Y_t = \beta_1 + \beta_2 Y_{t-1} + \left(\beta_3 \sin \frac{2\pi}{s} t + \beta_4 \cos \frac{2\pi}{s} t\right) + u_t,$$

where

- β_3, β_4 - additional model coefficients
- s - cycle frequency (in periods of time) as before

3. Dealing With Non-Stationarity

Quarterly Trend Example

- Suppose that we have **monthly data** and wish to include **quarterly trend** term
- Corresponding frequency is $s = 3$, and we estimate

$$Y_t = \beta_1 + \beta_2 Y_{t-1} + \left(\beta_3 \sin \frac{2\pi}{3} t + \beta_4 \cos \frac{2\pi}{3} t \right) + u_t$$

3. Dealing With Non-Stationarity

Annual Trend Example

- Suppose that we have **monthly data** and wish to include **annual trend** term
- Corresponding frequency is $s = 12$, and we estimate

$$Y_t = \beta_1 + \beta_2 Y_{t-1} + \left(\beta_3 \sin \frac{2\pi}{12} t + \beta_4 \cos \frac{2\pi}{12} t\right) + u_t$$

3. Dealing With Non-Stationarity

Monthly Trend Example

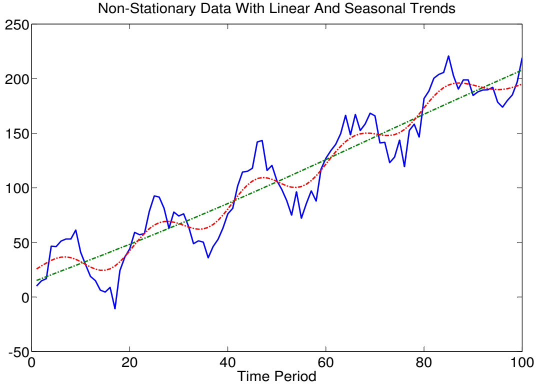
- Suppose that we have **daily data** and wish to include **monthly trend** term
- Corresponding frequency is $s = 30$, and we estimate

$$Y_t = \beta_1 + \beta_2 Y_{t-1} + \left(\beta_3 \sin \frac{2\pi}{30} t + \beta_4 \cos \frac{2\pi}{30} t\right) + u_t$$

3. Dealing With Non-Stationarity

Combining Linear, Quadratic and Seasonal Trends

- Some data may have a combination of trends



3. Dealing With Non-Stationarity

Combined Trends Through Differencing

- One solution is to apply **repeated differencing** to the series
- For example, first remove seasonal trend with **seasonal differences** $\Delta^s Y_t$
- Then, remove linear trend by taking **first or second difference**
$$\Delta^{1,s} Y_t = \Delta^s Y_t - \Delta^s Y_{t-1}$$
- Inspect model fit by examining residuals Q-Q plot

3. Dealing With Non-Stationarity

Combined Trends Through Trend Modelling

- Alternatively, **include both linear and cyclical trend terms** into the model

$$
\begin{aligned}
Y_t &= \beta_1 + \beta_2 Y_{t-1} & (1) \\
&+ \left(\beta_3 t + \beta_4 t^{\beta_5}\right) & (2) \\
&+ \left(\beta_6 \sin \frac{2\pi}{s} t + \beta_7 \cos \frac{2\pi}{s} t\right) & (3) \\
&+ u_t, & (4)
\end{aligned}
$$

where (1) is the $AR(1)$ part, (2) is the linear and quadratic trend terms, (3) is cyclical trend term and (4) is model error

4. Time-Series Forecasting

Forecasting Trends

- Suppose $\hat{\beta}_1$, $\hat{\beta}_2$,...,$\hat{\beta}_7$ are estimates from an $AR(1)$ model with linear, exponential and cyclical trends
- Then at some time T we can predict
  - **Linear Trend** as $\hat{L}_T = \hat{\beta}_3 T$
  - **Exponential Trend** as $\hat{E}_T = \hat{\beta}_4 T^{\hat{\beta}_5}$
  - **Cyclical Trend** as $\hat{C}_T = \hat{\beta}_6 \sin \frac{2\pi}{s} T + \hat{\beta}_7 \cos \frac{2\pi}{s} T$

4. Time-Series Forecasting

Seasonal Adjustments and "De-Trending"

- Data are often available in seasonally adjusted and/or "de-trended" form
- Objective is to remove all trends
- Approach is to estimate a model with trend components only

4. Time-Series Forecasting

Seasonal Adjustments and "De-Trending"

- For example, suppose data have exponential and cyclical trend components
- Estimate the trend-only model

$$Y_t = \alpha_1 t^{\alpha_2} + \left(\alpha_3 \sin \frac{2\pi}{s} t + \alpha_4 \cos \frac{2\pi}{s} t \right) + u_t$$

- Calculate the trend estimates
  - Exponential trend component $\hat{E}_t = \hat{\alpha}_1 t^{\hat{\alpha}_2}$
  - Cyclical trend component $\hat{C}_t = \hat{\alpha}_3 \sin \frac{2\pi}{s} t + \hat{\alpha}_4 \cos \frac{2\pi}{s} t$

4. Time-Series Forecasting

Seasonal Adjustments and "De-Trending"

- **De-trended data**: $\bar{Y}_t = Y_t - \hat{E}_t$
- **Seasonally-adjusted data:** $\tilde{Y}_t = Y_t - \hat{C}_t$

4. Time-Series Forecasting

Forecasting Series

- Given series value at time t, predict future value as $\hat{Y}_{t+1} = \hat{\beta}_1 + \hat{\beta}_2 Y_t + \hat{L}_{t+1} + \hat{E}_{t+1} + \hat{C}_{t+1}$
- **Question:** How to evaluate forecast accuracy?

4. Time-Series Forecasting

Forecast Evaluation: RMSE

- Consider **forecast error** $\hat{e}_t = Y_t - \hat{Y}_t$
- Fine **root mean squared error** as $RMSE = \sqrt{\frac{\sum_{t=1}^{T} \hat{e}_t^2}{T}}$
- Model with lowest RMSE may be preferred

4. Time-Series Forecasting

Forecast Evaluation: LINEX

- An alternative to RMSE is **LINEX loss function**

$$L_t(\hat{e}_t) = exp(-a\hat{e}_t) + a\hat{e}_t - 1$$

- $L_t(\hat{e}_t)$ is "penalty" for forecast error $\hat{e}_t$
- **Key feature**: positive and negative errors penalized differently
  - **Greater penalty for positive errors** when $a > 0$
  - **Greater penalty for negative errors** when $a < 0$
- Choose a based on the problem and select model with lowest $\sum L_t(\hat{e}_t)$

5. Summary

Topics in Applied Time Series

1. Non-stationary data may lead to biased estimates
2. Residual plots (Q-Q plot) may help detect non-stationarity
3. Several ways to account for non-stationarity
   - Differencing
   - Explicit trend modelling
4. RMSE, LINEX loss commonly used to gauge performance